A boundary value problem for nonlinear parabolic equations of the second order

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Introduction. We use the symbol t to denote a point on the real line $(-\infty, +\infty)$ and x to denote a point $(x_1, \cdots, x_m)$ in Euclidean m-space E^m. Let Ω be a domain (bounded or unbounded) in E^m with boundary $\dot{\Omega}$ and closure $\bar{\Omega}$. We put $Q_{t_1,t_2}=(t_1, t_2]\times\Omega$ and $\dot{Q}_{t_1,t_2}=(t_1, t_2]\times\dot{\Omega}$, where $t_1, t_2\in(-\infty, +\infty)$, $t_1<t_2$. Q_{t_1,t_2} is a cylinder in the $(m+1)$-dimensional space E^{m+1} of points (t, x) with Ω as its base, and $\dot{Q}_{t_1,t_2}$ is the lateral boundary of Q_{t_1,t_2}. If $t_1=-\infty$ and $t_2=+\infty$, then Q_{t_1,t_2} and $\dot{Q}_{t_1,t_2}$ are written simply as Q and $\dot{Q}$, respectively. $\bar{Q}_{t_1,t_2}(\bar{Q})$ denotes the closure of $Q_{t_1,t_2}(Q)$. The part of the boundary of Q_{t_1,t_2} which consists of the lateral boundary and the lower base is called the normal boundary of Q_{t_1,t_2} and is denoted by the symbol Γ_{t_1,t_2}. We denote by $Q_{t_1,t_2;R}$ the intersection of Q_{t_1,t_2} with the circular cylinder $C_R=(-\infty, +\infty)\times\{x: |x|<R\}$, and by $\dot{Q}_{t_1,t_2;R}$, $\Gamma_{t_1,t_2;R}$ the lateral boundary and the normal boundary of $Q_{t_1,t_2;R}$, respectively.

We write

$$u_t=\frac{\partial u}{\partial t}\,,\quad u_x=\left(\frac{\partial u}{\partial x_i}\right),\quad u_{xx}=\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)\qquad (i, j=1, \cdots, m)\,,$$

and similarly for other functions. We say that a function $u(t, x)$ is regular in $\bar{Q}_{t_1,t_2}$ if it is continuous in $\bar{Q}_{t_1,t_2}$ and if it has the derivative u_t and the continuous derivatives u_x, u_{xx} in Q_{t_1,t_2}. We say that a function $u(t, x)$ defined in $\bar{Q}_{t_1,t_2}$ (or on $\dot{Q}_{t_1,t_2}$) belongs to class $\bar{E}_1(K)$ there if Ω (or $\dot{\Omega}$) is unbounded and if there exist positive constants M and k $(k<K)$ such that

$$u(t, x)\leqq M\exp\left(k\sum_{i=1}^{m}|x_i|\right)\quad\text{in}\quad \bar{Q}_{t_1,t_2}\quad(\text{or on }\dot{Q}_{t_1,t_2})\,.$$

$u(t, x)$ is said to belong to class $\underline{E}_1(K)$ in $\bar{Q}_{t_1,t_2}$ (or on $\dot{Q}_{t_1,t_2}$) if there exist positive constants M and k $(k<K)$ such that

$$u(t, x)\geqq -M\exp\left(k\sum_{i=1}^{m}|x_i|\right)\quad\text{in}\quad \bar{Q}_{t_1,t_2}\quad(\text{or on }\dot{Q}_{t_1,t_2})\,.$$

We denote by $E_1(K)$ the class of functions belonging to $\bar{E}_1(K)$ and $\underline{E}_1(K)$ simultaneously.

Let $f(t, x, z, p, r)$ be a function defined for $(t, x)\in\bar{Q}$ and arbitrary z, $p=(p_i)$, $r=(r_{ij})$ $(i, j=1, \cdots, m)$, where p is an m-vector and r is a symmetric m by m matrix. We say following J. Szarski that the function

$f(t, x, z, p, r)$ is elliptic with respect to a function $u(t, x)$ having $u_x(t, x)$ in Q if for any symmetric matrices $r, \bar{r}$ such that $r \leqq \bar{r}$ (i.e. $\bar{r}-r$ is positive semi-definite) we get

$$f(t, x, u(t, x), u_x(t, x), r) \leqq f(t, x, u(t, x), u_x(t, x), \bar{r}) \quad \text{in} \quad Q .$$

If $r \leqq \bar{r}$ implies $f(t, x, z, p, r) \leqq f(t, x, z, p, \bar{r})$ for all t, x, z, p under consideration, then $f(t, x, z, p, r)$ is called simply elliptic.

We say that the function $f(t, x, z, p, r)$ satisfies the condition $(\mathfrak{M})$ if there is a positive constant L_0 such that

$$f(t, x, z, p, r)-f(t, x, \bar{z}, p, r) \leqq -L_0(z-\bar{z}) \quad \text{for} \quad z>\bar{z} ,$$

and that it satisfies the condition $(\mathfrak{L})$ if there are positive constants L_1, L_2 such that

$$|f(t, x, z, p, r)-f(t, x, z, \bar{p}, \bar{r})| \leqq L_1 \sum_{i=1}^{m} |p_i-\bar{p}_i| + L_2 \sum_{i,j=1}^{m} |r_{ij}-\bar{r}_{ij}| ,$$

where $p=(p_i)$, $\bar{p}=(\bar{p}_i)$, $r=(r_{ij})$, $\bar{r}=(\bar{r}_{ij})$ $(i, j=1, \cdots, m)$.

In this paper we shall be concerned with the nonlinear differential equation

$$\text{(A)} \qquad u_t=f(t, x, u, u_x, u_{xx}) ,$$

which is called parabolic when $f(t, x, z, p, r)$ is elliptic in the sense defined above. It is the purpose of this paper to discuss the existence and uniqueness of the solution to the first boundary problem (without initial condition!) for (A), which consists in searching for a function $u(t, x)$ satisfying the equation (A) in Q and the given boundary condition

$$\text{(B)} \qquad u(t, x)=\varphi(t, x) \quad \text{on} \quad \dot{Q} .$$

In Section 1 we shall consider the problem (A), (B) in the cylinder Q whose base Ω is bounded. Following I. I. Shmulev [4, 5] we shall prove theorems on the existence and uniqueness of bounded solutions. In Section 2 we shall be concerned with the problem (A), (B) in the cylinder Q whose base Ω is unbounded. We shall apply the method employed by P. Besala [1] in conjunction with that of I. I. Shmulev [4, 5] to establish theorems concerning the existence and uniqueness of solutions belonging to class $E_1(K)$ in $\bar{Q}$.

§1. In this section we shall consider the boundary problem (A), (B) in the cylinder Q whose base Ω is bounded.

(a) *Uniqueness of the solution.*

THEOREM 1.1. *Let the following conditions be satisfied:*

1° $f^{(1)}(t, x, z, p, r) \leqq f^{(2)}(t, x, z, p, r)$ *for* $(t, x) \in \bar{Q}$ *and arbitrary* z, p, r.

2° *At least one of* $f^{(1)}, f^{(2)}$ *satisfies the condition* $(\mathfrak{M})$.

3° $u^{(1)}(t, x)$ *and* $u^{(2)}(t, x)$ *are regular and bounded in* $\bar{Q}$.

4° $f^{(1)}$ *is elliptic with respect to* $u^{(1)}$, *or* $f^{(2)}$ *is elliptic with respect to* $u^{(2)}$.

5° *There hold the following inequalities:*

$$\begin{aligned} &u_t^{(1)}-f^{(1)}(t, x, u^{(1)}, u_x^{(1)}, u_{xx}^{(1)}) \\ &\leqq u_t^{(2)}-f^{(2)}(t, x, u^{(2)}, u_x^{(2)}, u_{xx}^{(2)}) \quad in \quad Q, \end{aligned} \tag{1.1}$$

$$u^{(1)}(t, x) \leqq u^{(2)}(t, x) \quad on \quad \dot{Q}. \tag{1.2}$$

Under these conditions we have

$$u^{(1)}(t, x) \leqq u^{(2)}(t, x) \quad in \quad \bar{Q}. \tag{1.3}$$

Proof. Assuming, for example, that (i) $f^{(1)}$ satisfies the condition $(\mathfrak{M})$ and (ii) $f^{(2)}$ is elliptic with respect to $u^{(2)}$, we shall derive the conclusion of the theorem. The remaining cases can be treated analogously.

We shall show that the inequality (1.3) holds in $\bar{Q}_{-\infty,\sigma}$ for an arbitrary $\sigma \in (-\infty, +\infty)$. To this end we put

$$v^{(k)}(t, x)=e^{\lambda(t-\sigma)}u^{(k)}(t, x) \ (k=1, 2),\ v(t, x)=v^{(1)}(t, x)-v^{(2)}(t, x)$$

and express (1.1) in terms of $v^{(k)}$ $(k=1, 2)$, v as follows:

$$\begin{aligned} &e^{-\lambda(t-\sigma)}(v_t-\lambda v) \\ &\leqq f^{(1)}(t, x, e^{-\lambda(t-\sigma)}v^{(2)}, e^{-\lambda(t-\sigma)}v_x^{(1)}, e^{-\lambda(t-\sigma)}v_{xx}^{(1)}) \\ &\quad -f^{(2)}(t, x, e^{-\lambda(t-\sigma)}v^{(2)}, e^{-\lambda(t-\sigma)}v_x^{(1)}, e^{-\lambda(t-\sigma)}v_{xx}^{(1)}) \\ &\quad +f^{(2)}(t, x, e^{-\lambda(t-\sigma)}v^{(2)}, e^{-\lambda(t-\sigma)}v_x^{(1)}, e^{-\lambda(t-\sigma)}v_{xx}^{(1)}) \\ &\quad -f^{(2)}(t, x, e^{-\lambda(t-\sigma)}v^{(2)}, e^{-\lambda(t-\sigma)}v_x^{(2)}, e^{-\lambda(t-\sigma)}v_{xx}^{(2)}) \\ &\quad +f^{(1)}(t, x, e^{-\lambda(t-\sigma)}v^{(1)}, e^{-\lambda(t-\sigma)}v_x^{(1)}, e^{-\lambda(t-\sigma)}v_{xx}^{(1)}) \\ &\quad -f^{(1)}(t, x, e^{-\lambda(t-\sigma)}v^{(2)}, e^{-\lambda(t-\sigma)}v_x^{(1)}, e^{-\lambda(t-\sigma)}v_{xx}^{(1)}). \end{aligned} \tag{1.4}$$

Let us take λ so that $0<\lambda<L_0$, L_0 being the constant of the condition $(\mathfrak{M})$ assumed for $f^{(1)}$. Since $v(t, x)=O(e^{\lambda(t-\sigma)})$ in $\bar{Q}_{-\infty,\sigma}$, there is a point $(t^\circ, x^\circ) \in \bar{Q}_{-\infty,\sigma}$ such that $v(t^\circ, x^\circ)=\sup_{\bar{Q}_{-\infty\,\sigma}} v(t, x)$. Suppose that $v(t^\circ, x^\circ)>0$. Then by (1.2) the point (t°, x°) cannot lie on the lateral boundary $\dot{Q}_{-\infty,\sigma}$ and there hold the relations:

$$v_t(t^\circ, x^\circ) \geqq 0, \quad v_x(t^\circ, x^\circ)=0, \quad v_{xx}(t^\circ, x^\circ) \leqq 0. \tag{1.5}$$

Substituting $t=t^\circ$, $x=x^\circ$ in (1.4), we find that by 1° the difference of the first two terms on the right-hand side of (1.4) is non-positive and that by the assumed ellipticity of $f^{(2)}$ the difference of the second two terms is also non-positive. Taking this into account and using the condition $(\mathfrak{M})$ assumed for $f^{(1)}$ we obtain from (1.4)

$$v_t(t^\circ, x^\circ) \leqq -(L_0-\lambda)v(t^\circ, x^\circ)<0,$$

which contradicts the first inequality of (1.5). Hence $v(t^\circ, x^\circ)$ cannot be positive. Consequently, the inequality (1.3) holds in $\bar{Q}_{-\infty,\sigma}$. In view of arbitrariness of σ (1.3) holds throughout $\bar{Q}$.

THEOREM 1.2. *If the function $f(t, x, z, p, r)$ is elliptic and satisfies the condition* ($\mathfrak{M}$) *and if the boundary function $\varphi(t, x)$ is continuous and bounded on $\dot{Q}$, then the first boundary problem* (A), (B) *has at most one bounded regular solution.*

This is an immediate consequence of Theorem 1.1.

(**b**) *Existence of the solution.*

Hypothesis (**I**). For any $t_1, t_2 \in (-\infty, +\infty)$, $t_1 < t_2$, and for any continuous function $\Psi(t, x)$ on Γ_{t_1, t_2} there exists a regular solution to the initial-boundary value problem

$$u_t = f(t, x, u, u_x, u_{xx}) \text{ in } Q_{t_1, t_2};\ u(t, x) = \Psi(t, x) \text{ on } \Gamma_{t_1, t_2}.$$

THEOREM 1.3. *Assume that:*

1° *$f(t, x, z, p, r)$ is elliptic;*
2° *$f(t, x, z, p, r)$ satisfies the condition* ($\mathfrak{M}$);
3° *$f(t, x, 0, 0, 0)$ is bounded in $\bar{Q}$;*
4° *$\varphi(t, x)$ is bounded and continuous on $\dot{Q}$;*
5° *the hypothesis* (I) *is satisfied.*

Then there exists a unique bounded regular solution to the boundary problem (A), (B). *If, in addition, f and φ are periodic* (*period T*) *in t* (*or time-independent*), *then so is the solution.*

REMARK. This theorem is a generalization of Theorem 1 of I. I. Shmulev [1]. The proof is based on the method applied by I. I. Shmulev [2] (see Theorem 1).

Proof. Let $\{s_n\}$, $\{t_n\}$ be sequences of numbers such that

$$0 > s_1 > s_2 > \cdots, s_n \to -\infty; \qquad 0 < t_1 < t_2 < \cdots, t_n \to +\infty.$$

For simplicity we put $Q_n = Q_{s_n, t_n}$ and $\dot{Q}_n = \dot{Q}_{s_n, t_n}$. We determine $u^n(t, x)$ to be the regular solution to the initial-boundary value problem:

$$\begin{aligned} &u_t^n = f(t, x, u^n, u_x^n, u_{xx}^n) \quad \text{in} \quad Q_n, \\ &u^n(t, x) = \varphi^n(t, x) \quad \text{on} \quad \dot{Q}_n, \quad u^n|_{t=s_n} = 0, \end{aligned} \tag{1.6}$$

where $\varphi^n(t, x) = \zeta^n(t)\varphi(t, x)$, $\zeta^n(t)$ being a continuous function such that $|\zeta^n(t)| \leq 1$, $\zeta^n(t) = 1$ for $t \geqq s_n + \delta$ ($\delta > 0$: small) and $\zeta^n(s_n) = 0$ ($n = 1, 2, \cdots$).

First note that the sequence $\{u^n(t, x)\}$ thus constructed according to Hypothesis (I) is uniformly bounded, namely:

(1.7) $$|u^n(t, x)| \leqq \max\left\{\sup_{\bar{Q}} \frac{|f(t, x, 0, 0, 0)|}{L_0}, \sup_{\dot{Q}} |\varphi(t, x)|\right\} \equiv M_0 \text{ in } \bar{Q}_n \qquad (n=1, 2, \cdots)$$

where L_0 is the constant of $(\mathfrak{M})$. We introduce the functions $v^n(t, x)$ defined by the relations:

(1.8) $$v^n(t, x) = e^{\lambda t} u^n(t, x), \quad 0 < \lambda < L_0 \qquad (n=1, 2, \cdots).$$

Put $u^{pq}(t, x) = u^p(t, x) - u^q(t, x)$, $v^{pq}(t, x) = v^p(t, x) - v^q(t, x)$, where p, q $(p<q)$ are natural numbers. By (1.6) $v^{pq}(t, x)$ satisfies the following relations:

(1.9) $$e^{-\lambda t}(v_t^{pq} - \lambda v^{pq}) = f(t, x, e^{-\lambda t} v^p, e^{-\lambda t} v_x^p, e^{-\lambda t} v_{xx}^p) - f(t, x, e^{-\lambda t} v^q, e^{-\lambda t} v_x^q, e^{-\lambda t} v_{xx}^q) \quad \text{in} \quad Q_{s_p+\delta, t_p},$$

(1.10) $$v^{pq}(t, x) = 0 \quad \text{on} \quad \dot{Q}_{s_p+\delta, t_p}, \qquad v^{pq}|_{t=s_p+\delta} = v^p|_{t=s_p+\delta} - v^q|_{t=s_p+\delta}.$$

Let $A_{pq} = \sup_{\bar{Q}_{s_p+\delta\, t_p}} |v^{pq}(t, x)|$ and let (t^{pq}, x^{pq}) be a point of $\bar{Q}_{s_p+\delta, t_p}$ such that $A_{pq} = |v^{pq}(t^{pq}, x^{pq})|$. If $(t^{pq}, x^{pq}) \in \Gamma_{s_p+\delta, t_p}$, then by (1.7), (1.8), (1.10) we get $A_{pq} \leqq 2M_0 e^{\lambda(s_p+\delta)}$. If (t^{pq}, x^{pq}) belongs to $\bar{Q}_{s_p+\delta, t_p} - \Gamma_{s_p+\delta, t_p}$, then $A_{pq} = 0$ by virtue of (1.9) and the assumptions 1°, 2° of the theorem. It follows therefore that

$$|v^{pq}(t, x)| \leqq 2M_0 e^{\lambda(s_p+\delta)} \quad \text{in} \quad \bar{Q}_{s_p+\delta, t_p},$$

whence

$$|u^{pq}(t, x)| \leqq 2M_0 e^{\lambda(-t+s_p+\delta)} \quad \text{in} \quad \bar{Q}_{s_p+\delta, t_p}.$$

This enables us to conclude that the sequence $\{u^n(t, x)\}$ is convergent locally uniformly in $\bar{Q}$. We denote by $u(t, x)$ the limit of this sequence. Clearly, $u(t, x)$ is bounded and continuous in $\bar{Q}$.

Let now $w(t, x)$ denote the regular solution to the problem:

$$w_t = f(t, x, w, w_x, w_{xx}) \quad \text{in} \quad Q_{s^*, t^*},$$
$$w(t, x) = \varphi(t, x) \quad \text{on} \quad \dot{Q}_{s^*, t^*}, \quad w|_{t=s^*} = u(s^*, x),$$

where s^*, t^* $(s^* < t^*)$ are arbitrary fixed numbers. Observing that the function $w^n(t, x) = w(t, x) - u^n(t, x)$ for sufficiently large n satisfies the equation

$$w_t^n = f(t, x, w, w_x, w_{xx}) - f(t, x, u^n, u_x^n, u_{xx}^n) \quad \text{in} \quad Q_{s^*, t^*},$$

and the initial-boundary condition

$$w^n(t, x) = 0 \quad \text{on} \quad \dot{Q}_{s^*, t^*}, \quad w^n|_{t=s^*} = u(s^*, x) - u^n(s^*, x),$$

we can show that

(1.11) $$|w(t, x) - u^n(t, x)| \leqq \max_{\bar{\Omega}} |u(s^*, x) - u^n(s^*, x)| \quad \text{in} \quad \bar{Q}_{s^*, t^*}.$$

Since the right member of (1.11) tends to zero as $n \to +\infty$, it follows from (1.11) that $u(t, x)$ coincides with $w(t, x)$ in $\bar{Q}_{s^*, t^*}$. In view of arbitrariness of s^*, t^* we conclude that the function $u(t, x)$ constitutes the regular solution sought to the boundary problem (A), (B).

The proof of the statement concerning periodicity (or time-independence) is immediate.

§ 2. In this section we shall consider the boundary problem (A), (B) in the cylinder Q whose base Ω is unbounded.

(a) *Uniqueness of the solution.*

THEOREM 2.1. *Let the following assumptions be satisfied:*

1° $f^{(1)}(t, x, z, p, r) \leqq f^{(2)}(t, x, z, p, r)$ *for* $(t, x) \in \bar{Q}$ *and arbitrary* z, p, r.

2° *At least one of* $f^{(1)}, f^{(2)}$ *satisfies the condition* $(\mathfrak{M})$, *and at least one of them satisfies the condition* $(\mathfrak{L})$. *We denote by* L_0 *the constant of the condition* $(\mathfrak{M})$, *and by* L_1, L_2 *the constants of the condition* $(\mathfrak{L})$.

3° $u^{(1)}(t, x)$ *and* $u^{(2)}(t, x)$ *are regular in* $\bar{Q}$ *and belong to* $\bar{E}_1(K)$ *and* $\underline{E}_1(K)$, *respectively, where* K *is the positive root of the equation in* ξ

$$m^2 L_2 \xi^2 + m L_1 \xi - L_0 = 0 \; . \tag{2.1}$$

4° $f^{(1)}$ *is elliptic with respect to* $u^{(1)}$, *or* $f^{(2)}$ *is elliptic with respect to* $u^{(2)}$.

5° *There hold the following inequalities:*

$$\begin{aligned} &u_t^{(1)} - f^{(1)}(t, x, u^{(1)}, u_x^{(1)}, u_{xx}^{(1)}) \\ &\qquad \leqq u_t^{(2)} - f^{(2)}(t, x, u^{(2)}, u_x^{(2)}, u_{xx}^{(2)}) \quad in \quad Q \; , \end{aligned} \tag{2.2}$$

$$u^{(1)}(t, x) \leqq u^{(2)}(t, x) \quad on \quad \dot{Q} \; . \tag{2.3}$$

Under these assumptions we have

$$u^{(1)}(t, x) \leqq u^{(2)}(t, x) \quad in \quad \bar{Q} \; . \tag{2.4}$$

REMARK. This theorem is formulated similarly to Theorem 1 of P. Besala [2], and generalizes Theorem 1 of M. Kono [3]. The proof of this theorem is similar to that of Theorem II of P. Besala [1].

Proof. We discuss the case in which (i) $f^{(1)}$ satisfies the condition $(\mathfrak{L})$, (ii) $f^{(2)}$ satisfies the condition $(\mathfrak{M})$ and (iii) $f^{(1)}$ is elliptic with respect to $u^{(1)}$. The remaining cases may be dealt with quite analogously.

As in the proof of Theorem 1.1. it is sufficient to prove (2.4) in $\bar{Q}_{-\infty, \sigma}$, σ being an arbitrary fixed number in $(-\infty, +\infty)$. We introduce the auxiliary function

$$H(x; \lambda) = \prod_{i=1}^{m} \cosh \lambda x_i \; , \tag{2.5}$$

where λ is a positive parameter. The following properties of $H(x; \lambda)$ will be needed later on:

(i) $2^{-m}\exp\left(\lambda\sum_{i=1}^{m}|x_i|\right)<H(x;\lambda)<\exp\left(\lambda\sum_{i=1}^{m}|x_i|\right)$;

(ii) If $\lambda<\lambda'$, then $\dfrac{H(x;\lambda)}{H(x;\lambda')}\to 0$ as $|x|\to+\infty$;

(iii) to each $\lambda(0<\lambda<K$, K being the positive root of (2.1)) there is associated a positive number $\delta(\lambda)$ such that

$$\mathcal{L}H\equiv L_2\sum_{i,j=1}^{m}\left|\frac{\partial^2 H}{\partial x_i\partial x_j}\right|+L_1\sum_{i=1}^{m}\left|\frac{\partial H}{\partial x_i}\right|-L_0H\leqq-\delta(\lambda)H\,. \tag{2.6}$$

By the assumption 3° there exist positive constants M_0 and λ_0 ($\lambda_0<K$) such that

$$u^{(1)}(t,x)-u^{(2)}(t,x)\leqq M_0\exp\left(\lambda_0\sum_{i=1}^{m}|x_i|\right)\quad\text{in}\quad\bar{Q}_{-\infty,\sigma}\,. \tag{2.7}$$

We put

$$\begin{aligned}v^{(k)}(t,x)&=e^{\mu(t-\sigma)}\frac{u^{(k)}(t,x)}{H(x;\lambda)}\qquad(k=1,2)\,,\\ v(t,x)&=v^{(1)}(t,x)-v^{(2)}(t,x)\,,\end{aligned} \tag{2.8}$$

where $\lambda_0<\lambda<K$ and $0<\mu<\delta(\lambda)$, and show that $v(t,x)\leqq 0$ in $\bar{Q}_{-\infty,\sigma}$. For this purpose we write the inequality (2.2) in terms of $v^{(k)}$ ($k=1,2$), v as follows:

$$\begin{aligned}&e^{-\mu(t-\sigma)}H(v_t-\mu v)\\ &\leqq f^{(1)}(t,x,e^{-\mu(t-\sigma)}Hv^{(1)},e^{-\mu(t-\sigma)}\tau^{(1)},e^{-\mu(t-\sigma)}\rho^{(1,1)})\\ &-f^{(1)}(t,x,e^{-\mu(t-\sigma)}Hv^{(1)},e^{-\mu(t-\sigma)}\tau^{(1)},e^{-\mu(t-\sigma)}\rho^{(2,1)})\\ &+f^{(1)}(t,x,e^{-\mu(t-\sigma)}Hv^{(1)},e^{-\mu(t-\sigma)}\tau^{(1)},e^{-\mu(t-\sigma)}\rho^{(2,1)})\\ &-f^{(1)}(t,x,e^{-\mu(t-\sigma)}Hv^{(1)},e^{-\mu(t-\sigma)}\tau^{(2)},e^{-\mu(t-\sigma)}\rho^{(2,2)})\\ &+f^{(1)}(t,x,e^{-\mu(t-\sigma)}Hv^{(1)},e^{-\mu(t-\sigma)}\tau^{(2)},e^{-\mu(t-\sigma)}\rho^{(2,2)})\\ &-f^{(2)}(t,x,e^{-\mu(t-\sigma)}Hv^{(1)},e^{-\mu(t-\sigma)}\tau^{(2)},e^{-\mu(t-\sigma)}\rho^{(2,2)})\\ &+f^{(2)}(t,x,e^{-\mu(t-\sigma)}Hv^{(1)},e^{-\mu(t-\sigma)}\tau^{(2)},e^{-\mu(t-\sigma)}\rho^{(2,2)})\\ &-f^{(2)}(t,x,e^{-\mu(t-\sigma)}Hv^{(2)},e^{-\mu(t-\sigma)}\tau^{(2)},e^{-\mu(t-\sigma)}\rho^{(2,2)})\,,\end{aligned} \tag{2.9}$$

where we have set

$$\begin{aligned}\tau^{(k)}&=\left(\frac{\partial v^{(k)}}{\partial x_i}H+v^{(k)}\frac{\partial H}{\partial x_i}\right)\\ \rho^{(k,h)}&=\left(\frac{\partial^2 v^{(k)}}{\partial x_i\partial x_j}H+\frac{\partial v^{(h)}}{\partial x_i}\frac{\partial H}{\partial x_j}+\frac{\partial v^{(h)}}{\partial x_j}\frac{\partial H}{\partial x_i}+v^{(h)}\frac{\partial^2 H}{\partial x_i\partial x_j}\right)\\ &\qquad(i,j=1,\cdots,m;\,k,h=1,2)\,.\end{aligned} \tag{2.10}$$

Let us now take an increasing sequence $\{R_n\}$, $R_n>0$, $R_n\to+\infty$ as $n\to+\infty$, and denote by Q_n, $\dot{Q}_n$ the cylinder $Q_{-\infty,\sigma;R_n}$ and its lateral boundary $\dot{Q}_{-\infty,\sigma;R_n}$, respectively. The lateral boundary $\dot{Q}_n$ of Q_n consists of two parts: the part $S_n^{(1)}=\dot{Q}_{-\infty,\sigma}\cap C_n$, $C_n=(-\infty,+\infty)\times\{x: |x|\leqq R_n\}$, and the remaining part $S_n^{(2)}$.

Let $A_n=\sup_{\bar{Q}_n} v(t, x)$. There exists a point $(t^n, x^n)\in\bar{Q}_n$ such that $A_n=v(t^n, x^n)$. If $(t^n, x^n)\in S_n^{(1)}$, then $A_n\leqq 0$ by (2.3). If $(t^n, x^n)\in S_n^{(2)}$, then from (2.7), (2.8) and the property (i) of $H(x;\lambda)$ we obtain $A_n\leqq 2^m M_0 e^{-(\lambda-\lambda_0)R_n}$. Finally let $(t^n, x^n)\in\bar{Q}_n-\dot{Q}_n$. Then we have

$$v_t(t^n, x^n)\geqq 0\ ,\quad v_x(t^n, x^n)=0\ ,\quad v_{xx}(t^n, x^n)\leqq 0\ . \tag{2.11}$$

Assume that $A_n>0$ and substitute $t=t^n$, $x=x^n$ in (2.9). Since $f^{(1)}$ is elliptic with respect to $u^{(1)}$, the difference of the first two terms on the right-hand side of (2.9) is non-positive. By 1° the difference of the third two terms is also non-positive. Taking this into account and estimating the remaining terms by means of the conditions ($\mathfrak{M}$), ($\mathfrak{L}$) we conclude that

$$\begin{aligned} v_t H &\leqq v\Big(L_2\sum_{i,j=1}^{m}\Big|\frac{\partial^2 H}{\partial x_i\partial x_j}\Big| + L_1\sum_{i=1}^{m}\Big|\frac{\partial H}{\partial x_i}\Big| - L_0 H+\mu H\Big)\\ &= v(\mathcal{L}H+\mu H)\leqq -(\delta(\lambda)-\mu)vH<0 \quad\text{at}\quad (t^n, x^n)\ ,\end{aligned}$$

which contradicts the first inequality of (2.11). It is thus shown that $(t^n, x^n)\in\bar{Q}_n-\dot{Q}_n$ implies $A_n\leqq 0$. We have thus established the estimate:

$$v(t, x)\leqq 2^m M_0 e^{-(\lambda-\lambda_0)R_n}\quad\text{in}\quad \bar{Q}_n\ . \tag{2.12}$$

Given an arbitrary small number $\varepsilon>0$ and an arbitrary point (t^*, x^*) of $\bar{Q}_{-\infty,\sigma}$, we can take n so large that $(t^*, x^*)\in\bar{Q}_n$ and at the same time that the right-hand side of (2.12) is less than ε. Then $v(t^*, x^*)<\varepsilon$, which implies that $v(t, x)\leqq 0$ in $\bar{Q}_{-\infty,\sigma}$, or equivalently, $u^{(1)}(t, x)\leqq u^{(2)}(t, x)$ in $\bar{Q}_{-\infty,\sigma}$.

The following uniqueness theorem follows from Theorem 2.1.

THEOREM 2.2. *If the function $f(t, x, z, p, r)$ is elliptic and satisfies the conditions ($\mathfrak{M}$) and ($\mathfrak{L}$) and if the boundary function $\varphi(t, x)$ is continuous and of class $E_1(K)$ on $\dot{Q}$, where K is the positive root of* (2.1), *then there exists at most one solution to the problem* (A), (B) *which is regular and of class $E_1(K)$ in $\bar{Q}$.*

(**b**) *Existence of the solution.*

THEOREM 2.3. *Assume that:*

1° *$f(t, x, z, p, r)$ is elliptic;*

2° *$f(t, x, z, p, r)$ satisfies the conditions* ($\mathfrak{M}$) *and* ($\mathfrak{L}$);

3° *$f(t, x, 0, 0, 0)$ belongs to class $E_1(K)$ in $\bar{Q}$, where K is the positive root of the equation* (2.1);

4° *$\varphi(t, x)$ is continuous and of class $E_1(K)$ in $\bar{Q}$;*

5° *the following hypothesis is fulfilled:*

Hypothesis (II). *For any* $t_1, t_2 \in (-\infty, +\infty)$, $t_1 < t_2$, $R > 0$ *and for any continuous function* $\Phi(t, x)$ *of class* $E_1(K)$ *in* $\bar{Q}$ *there exists a regular solution to the first initial-boundary value problem:*

$$u_t = f(t, x, u, u_x, u_{xx}) \quad in \quad Q_{t_1, t_2; R}\,,$$
$$u(t, x) = \Phi(t, x) \quad on \quad \Gamma_{t_1, t_2; R}\,.$$

Then there exists a unique solution to the problem (A), (B) *which is regular and of class* $E_1(K)$ *in* $\bar{Q}$. *If, in addition,* f *and* φ *are periodic (period* T*) in* t *(or time-independent), then so is the solution.*

REMARK. The proof of this theorem is based on the method employed by P. Besala [1] (Theorem III) combined with that of I. I. Shmulev [4, 5]. This theorem generalizes Theorem 2 of M. Kono [3].

Proof. First of all note that there are positive constants M_0 and λ_0 ($\lambda_0 < K$) such that

$$(2.13) \qquad |f(t, x, 0, 0, 0)|\,, \quad |\varphi(t, x)| \leqq M_0 \exp\left(\lambda_0 \sum_{i=1}^{m} |x_i|\right) \quad \text{in} \quad \bar{Q}\,.$$

Let $\{s_n\}$, $\{R_n\}$ be sequences of numbers such that

$$0 > s_1 > s_2 \cdots, \; s_n \to -\infty\,; \quad 0 < R_1 < R_2 < \cdots, \; R_n \to +\infty\,.$$

We set $Q_n = Q_{s_n, 0; R_n}$ and $\Gamma_n = \Gamma_{s_n, 0; R_n}$. The lateral boundary $\dot{Q}_n$ of Q_n consists of two parts: the part $S_n^{(1)} = \dot{Q}_n \cap C_n$, $C_n = (-\infty, +\infty) \times \{x : |x| \leqq R_n\}$ and the remaining part $S_n^{(2)}$.

Let us construct the sequence of functions $\{u^n(t, x)\}$ by solving according to Hypothesis (II) the initial-boundary problems:

$$(2.14) \qquad u_t^n = f(t, x, u^n, u_x^n, u_{xx}^n) \quad \text{in} \quad Q_n\,,$$

$$(2.15) \qquad u^n(t, x) = \varphi(t, x) \quad \text{on} \quad \Gamma_n \qquad (n = 1, 2, \cdots)\,.$$

We shall prove that this sequence converges in $\bar{Q}_{-\infty, 0}$ to a function $u^{(0)}(t, x)$ satisfying

$$(2.16) \qquad \begin{aligned} &u_t^{(0)} = f(t, x, u^{(0)}, u_x^{(0)}, u_{xx}^{(0)}) \quad \text{in} \quad Q_{-\infty, 0} \\ &u^{(0)}(t, x) = \varphi(t, x) \quad \text{on} \quad \dot{Q}_{-\infty, 0}\,. \end{aligned}$$

For this purpose we put

$$(2.17) \qquad v^n(t, x) = \frac{u^n(t, x)}{H(x; \lambda')} \qquad (n = 1, 2, \cdots)\,,$$

where H is defined by (2.5) and λ' is such that $\lambda_0 < \lambda' < K$, and write the the equation that v^n satisfies as follows:

(2.18)

$$v_t^n H = f\left(t, x, v^n H, \frac{\partial v^n}{\partial x_i} H + v^n \frac{\partial H}{\partial x_i}, \frac{\partial^2 v^n}{\partial x_i \partial x_j} H + \frac{\partial v^n}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{\partial v^n}{\partial x_j}\frac{\partial H}{\partial x_i} + v^n \frac{\partial^2 H}{\partial x_i \partial x_j}\right)$$

$$-f\left(t, x, v^n H, \frac{\partial v^n}{\partial x_i} H + v^n \frac{\partial H}{\partial x_i}, \frac{\partial v^n}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{\partial v^n}{\partial x_j}\frac{\partial H}{\partial x_i} + v^n \frac{\partial^2 H}{\partial x_i \partial x_j}\right)$$
$$+f\left(t, x, v^n H, \frac{\partial v^n}{\partial x_i} H + v^n \frac{\partial H}{\partial x_i}, \frac{\partial v^n}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{\partial v^n}{\partial x_j}\frac{\partial H}{\partial x_i} + v^n \frac{\partial^2 H}{\partial x_i \partial x_j}\right)$$
$$-f(t, x, 0, 0, 0) + f(t, x, 0, 0, 0) .$$

Let $A_n = \sup_{\bar{Q}_n} |v^n(t, x)|$ and let (t^n, x^n) be a point of $\bar{Q}_n$ such that $A_n = |v^n(t^n, x^n)|$. If $(t^n, x^n) \in \Gamma_n$, then by (2.15), (2.17), (2.13) and the property (i) of $H(x; \lambda)$ we have $A_n \leqq 2^m M_0$. Suppose that $(t^n, x^n) \in \bar{Q}_n - \Gamma_n$. If $v^n(t^n, x^n) \geqq 0$, then (2.11) holds. Substituting $t = t^n$, $x = x^n$ in (2.18), we see that the first difference on the right-hand side of (2.18) is non-positive by virtue of the ellipticity of f. Estimating the remaining terms by means of the conditions ($\mathfrak{M}$), ($\mathfrak{L}$) we then have at (t^n, x^n)

$$0 \leqq v_t^n \leqq v^n \frac{\mathcal{L}H}{H} + \frac{1}{H} f(t, x, 0, 0, 0) ,$$

whence, in view of (2.13) and the property (iii) of $H(x; \lambda)$, we have $A_n \leqq \frac{2^m M_0}{\delta(\lambda')}$. If $v^n(t^n, x^n) < 0$, then similarly we have the same inequality. We have thus shown that

$$|v^n(t, x)| \leqq \left\{1 + \frac{1}{\delta(\lambda')}\right\} 2^m M_0 \equiv M_1 \quad \text{in} \quad \bar{Q}_n . \tag{2.19}$$

Next we introduce the functions $w^n(t, x)$ defined by

$$w^n(t, x) = e^{\mu t} \frac{u^n(t, x)}{H(x; \lambda'')} \qquad (n = 1, 2, \cdots) , \tag{2.20}$$

where $\lambda' < \lambda'' < K$ and $0 < \mu < \delta(\lambda'')$. By (2.20), (2.17) we have

$$w^n(t, x) = e^{\mu t} \frac{H(x; \lambda')}{H(x; \lambda'')} v^n(t, x) . \tag{2.21}$$

We put $u^{pq} = u^p - u^q$, $v^{pq} = v^p - v^q$, $w^{pq} = w^p - w^q$, where p, q $(p < q)$ are natura numbers. The function w^{pq} satisfies in Q_p the equation

$$\begin{aligned} &e^{-\mu t} H(w_t^{pq} - \mu w^{pq}) \\ &= f(t, x, e^{-\mu t} H w^p, e^{-\mu t} \tau^{(p)}, e^{-\mu t} \rho^{(p, p)}) \\ &\quad - f(t, x, e^{-\mu t} H w^p, e^{-\mu t} \tau^{(p)}, e^{-\mu t} \rho^{(q, p)}) \\ &\quad + f(t, x, e^{-\mu t} H w^p, e^{-\mu t} \tau^{(p)}, e^{-\mu t} \rho^{(q, p)}) \\ &\quad - f(t, x, e^{-\mu t} H w^q, e^{-\mu t} \tau^{(q)}, e^{-\mu t} \rho^{(q, q)}) , \end{aligned} \tag{2.22}$$

where

$$\tau^{(k)} = \left(\frac{\partial w^k}{\partial x_i} H + w^k \frac{\partial H}{\partial x_i}\right) ,$$

$$\rho^{(k,h)} = \left(\frac{\partial^2 w^k}{\partial x_i \partial x_j} H + \frac{\partial w^h}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{\partial w^h}{\partial x_j}\frac{\partial H}{\partial x_i} + w^h \frac{\partial^2 H}{\partial x_i \partial x_j}\right)$$
$$(i, j=1, \cdots, m; k, h=p, q) .$$

Put $A_{pq}=\sup_{\bar{Q}_p} |w^{pq}(t, x)|$. There exists a point $(t^{pq}, x^{pq}) \in \bar{Q}_p$ such that $A_{pq}=|w^{pq}(t^{pq}, x^{pq})|$. If $(t^{pq}, x^{pq}) \in S_p^{(1)}$, then by (2.15) $A_{pq}=0$. If $(t^{pq}, x^{pq}) \in S_p^{(2)}$, then from (2.21), (2.19) and the property (i) of $H(x; \lambda)$ it follows that $A_{pq} \leqq 2M_1 e^{-(\lambda''-\lambda')R_p}$. If (t^{pq}, x^{pq}) is found on the lower base of Q_p, then $A_{pq} \leqq 2M_1 e^{\mu s_p}$. Assume finally that (t^{pq}, x^{pq}) belongs to $\bar{Q}_p - \Gamma_p$. Then, if $w^{pq}(t^{pq}, x^{pq})>0$, observing that the first difference on the right-hand side of (2.22) is non-positive by the ellipticity of f and estimating the second difference by means of the conditions ($\mathfrak{M}$), ($\mathfrak{L}$), we obtain the contradiction:

$$0 \leqq w_t^{pq} H \leqq w^{pq}(\mathcal{L}H+\mu H) \leqq -w^{pq}(\delta(\lambda'')-\mu)H<0 \quad \text{at} \quad (t^{pq}, x^{pq}) .$$

In an analogous manner we can derive a contradiction from the assumption $w^{pq}(t^{pq}, x^{pq})<0$, thereby proving that $A_{pq}=0$. We have thus arrived at the inequality:

$$(2.23) \qquad |w^{pq}(t, x)| \leqq 2M_1 \max\{e^{\mu s_p}, e^{-(\lambda''-\lambda')R_p}\} \quad \text{in} \quad \bar{Q}_p .$$

Let Δ be an arbitrary bounded and closed domain in $\bar{Q}_{-\infty,0}$. We choose p so large that $\Delta \subset \bar{Q}_p$. It then follows from (2.20) and (2.23) that

$$(2.24) \qquad |u^{pq}(t, x)| \leqq 2M_1 N_1 \max\{e^{\mu s_p}, e^{-(\lambda''-\lambda')R_p}\} \quad \text{in} \quad \Delta ,$$

where $N_1=\sup_{\Delta} H(x; \lambda'')$. Noting that the right-hand side of (2.24) can be made arbitrarily small by choosing p sufficiently large, we conclude that the sequence $\{u^n(t, x)\}$ is convergent locally uniformly in $\bar{Q}_{-\infty,0}$. Observe that the limit function $u^{(0)}(t, x)$ of this sequence belongs to class $E_1(K)$. In fact, from (2.17), (2.19) we get

$$(2.25) \qquad |u^{(0)}(t, x)| \leqq M_1 H(x; \lambda') \leqq M_1 \exp\left(\lambda' \sum_{i=1}^{m} |x_i|\right) \quad \text{in} \quad \bar{Q}_{-\infty,0} .$$

In order to prove that $u^{(0)}(t, x)$ satisfies (2.16) we need only show that, for any fixed p, the regular solution $u^*(t, x)$ to the initial-boundary value problem

$$u_t^* = f(t, x, u^*, u_x^*, u_{xx}^*) \quad \text{in} \quad Q_p , \qquad u^*(t, x)=u^{(0)}(t, x) \quad \text{on} \quad \Gamma_p$$

coincides with $u^{(0)}(t, x)$ in $\bar{Q}_p$. Given an arbitrary $\varepsilon>0$, there is an n_0 such that for $n>n_0$ the inequality

$$(2.26) \qquad |u^*(t, x)-u^n(t, x)|<\varepsilon \quad \text{on} \quad \Gamma_p$$

holds. The functions

$$w^*(t, x)=e^{\mu t}\frac{u^*(t, x)}{H(x; \lambda'')}, \quad w^n(t, x)=e^{\mu t}\frac{u^n(t, x)}{H(x; \lambda'')},$$
$$w^{*n}(t, x)=w^*(t, x)-w^n(t, x) \tag{2.27}$$

satisfy an equation similar to (2.22) in Q_p and the inequality $|w^{*n}(t, x)|<\varepsilon$ on Γ_p. Since, arguing as before, we can assert that $w^{*n}(t, x)$ takes on its positive maximum and negative minimum (in $\bar{Q}_p$), if any, on the normal boundary of Q_p, we obtain

$$|w^{*n}(t, x)|<\varepsilon \quad \text{in} \quad \bar{Q}_p,$$

whence in view of (2.27)

$$|u^*(t, x)-u^n(t, x)|<\varepsilon N \quad \text{in} \quad \bar{Q}_p,$$

whence $N=\sup_{\bar{Q}_p} e^{-\mu t}H(x; \lambda'')$. This means that $u^*(t, x)\equiv u^{(0)}(t, x)$ in $\bar{Q}_p$.

Let $\{t_n\}$ be an increasing sequence, $t_n>0$, $t_n\to+\infty$ as $n\to+\infty$. Proceeding entirely as before, we can construct, for every t_n, the regular solution $u^{(n)}(t, x)$ of class $E_1(K)$ to the boundary problem

$$u_t^{(n)}=f(t, x, u^{(n)}, u_x^{(n)}, u_{xx}^{(n)}) \quad \text{in} \quad \bar{Q}_{-\infty, t_n},$$
$$u^{(n)}(t, x)=\varphi(t, x) \quad \text{on} \quad \dot{Q}_{-\infty, t_n}.$$

The required solution is obtained by taking limit of the sequence $\{u^{(n)}(t, x)\}$ thus constructed, since it can be shown as in the proof of Theorem 2.1 that $u^{(n+1)}(t, x)$ coincides with $u^{(n)}(t, x)$ in $\bar{Q}_{-\infty, t_n}$ for every n.

The proof of the statement concerning periodicity (or time-independence) is immediate.

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